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Dynamics of Constrained Systems

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The dynamics on the final constraint submanifold of a constrained system is discussed: a system of equations of motion is induced by the initial ones, whose consistency originates this submanifold, and there is also a dynamics which is the geometrically natural one of this (presymplectic) submanifold. Their compatibility and possible equivalence are analyzed. As a consequence, another justification of Dirac's conjecture is obtained.

It is well known that one of the essential characteristics of constrained dynamical systems is that, in general, the final phase space (i.e., the set of the dynamical states) is not the initial phase space, but a submanifold of it (Dirac, 1964; Gotay *et al.*, 1978). As several authors have shown (Lichnerowicz, 1975; Gotay *et al.*, 1978; Gomis *et al.*, 1984; Cariñena *et al.*, 1985), this submanifold admits several kinds of alternative equations of motion which are compatible with the starting dynamics. The aim of this paper is to compare some of these alternatives, discussing their possible equivalence and its consequences.

In the Hamiltonian as well as in the Lagrangian formulation of mechanics for nonregular systems, the initial phase space is a *presymplectic manifold* (M_0, ω_0) [i.e., $\omega_0 \in Z^2(M_0)$ is a degenerate form of constant rank, where $Z^p(M_0)$ denotes the set of closed p-forms in M_0]. This M_0 can be a submanifold of an ambient symplectic manifold, as happens, for instance, in the Hamiltonian formulation of Dirac's theory of constrained systems (Dirac, 1964), where j_0: $M_0 \hookrightarrow T^*Q$ (j_0 is an imbedding) and M_0 is called the *primary constraint submanifold*, where $\omega_0 = j_0^*\Omega$ (Ω denotes the natural symplectic structure of T^*Q).

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The equations of motion are obtained by generalizing the regular case. Thus, from $\alpha_0 \in Z^1(M_0)$ (rank $\alpha_0 = \text{const}$), we write in compact notation

$$i(X_0)\omega_0 = \alpha_0; \qquad X_0 \in \chi(M_0) \tag{1}$$

where $i(X_0)\omega_0$ means the contraction of X_0 and ω_0. Here α_0 may be locally expressed as $\alpha_0 = dh_0$, with $h_0 \in C^\infty(M_0)$. The triad $(M_0, \omega_0, \alpha_0)$ is called a *presymplectic, locally Hamiltonian system* (plHs), with *locally Hamiltonian canonical function* h_0.

A first consequence of the degeneracy of ω_0 is that, in general, equations (1) are not compatible everywhere in M_0. This forces us to use algorithmic procedures in order to determine the submanifold of M_0 where such equations are compatible and have consistent solutions (Dirac, 1964; Gotay *et al.*, 1978). If such a submanifold $\hat{\imath}$: $C \hookrightarrow M_0$ exists, it is called the *final constraint submanifold* (fcs) and the equations

$$(i(X_0)\omega_0 - \alpha_0)|_C = 0 \tag{2}$$

are compatible and have consistent solutions $X_0 \in \underline{\chi(C)}$, where $\underline{\chi(C)}$ is the set of vector fields of $\chi(M_0)$ tangent to C. The fcs $\overline{(C, \omega_C)}$ is presymplectic in general, and $\omega_C = \hat{\imath}^*\omega_0$. Then, any triad (P, C, Ω) such that j: $C \hookrightarrow P$ and (P, Ω) is a symplectic manifold is called a *canonical system* associated to the plHs $(M_0, \omega_0, \alpha_0)$ (Sniatycki, 1974; Cariñena *et al.*, 1985) (the existence of such a symplectic manifold (P, Ω) is always assured for every presymplectic manifold (Gotay, 1982; Marle, 1983).

Degeneracy of ω_0 has another consequence: nonuniqueness of the solutions of (2). In fact, if X_0 is a solution, then $X_0 + Z$ is another possible solution $\forall Z \in \operatorname{Ker} \omega_0 \cap \chi(C) \equiv G^0$. Following the terminology of Gotay and Nester (1979) and Bergvelt and de Kerf (1986), we call the solutions of (2) *gauge equivalent vector fields*, and all the points in the fcs that are reached starting from the same initial condition by means of integral curves of these fields, in the same lapsus of the evolution parameter, are *gauge equivalent points* (and it is assumed they represent dynamically equivalent physical states). Finally, the vector fields whose integral curves are made up by gauge equivalent points are called *gauge vector fields* (gvf), and it is obvious that they must be tangent to C.

Although all the elements of G^0 are gvf, they do not exhaust the set of gvf. There exists an algorithmic procedure (Gotay and Nester, 1979; Bergvelt and de Kerf, 1986), which, starting from G^0, leads to a final set G which contains all the gvf of the theory. Obviously $G \subset \chi(C)$.

Equations (2) are relations "at support" on C; that is, they only hold on the points belonging to the submanifold C of M_0 (sometimes, in physical language, they are called "weak equalities" on C). These equations are induced on C by the plHs $(M_0, \omega_0, \alpha_0)$. Nevertheless, other dynamics (which are the natural ones of the fcs because they are exclusively related to the geometry of the fcs) may be independently defined in C. In fact, since ω_C is presymplectic in general, giving a closed 1-form $\beta_0 \in Z^1(C)$, we can stablish equations of motion in C in the following way:

$$i(X_C)\omega_C = \beta_C; \qquad X_C \in \chi(C) \tag{3}$$

with $\beta_C = dh_C$ locally for any $h_C \in C^\infty(C)$. The necessary and sufficient condition for this system to be compatible is $L(\operatorname{Ker}\omega_C)\beta_C = 0$ (where L denotes the Lie derivative). These equations can be related to the system (M_0, C, ω_0) (or, when appropriate, to any canonical system (P, C, Ω) associated to the plHs by introducing vector fields $X_0 \in \chi(M_0)$ such that $\hat{\imath}_* X_C = X_0|_C$ [resp. $X \in \chi(C)$ such that $j_* X_C = X|_C$] and 1-forms $\beta_0 \in \Lambda^1(M_0)$ such that $\hat{\imath}^*\beta_0 = \beta_C$ [resp. $\beta \in \Lambda^1(P)$ such that $j^*\beta = \beta_C$]. Therefore (3) may be written alternatively as

$$\hat{\imath}^*(i(X_0)\omega_0 - \beta_0) = 0 \qquad [\text{locally } \beta_0 = dh_0 \text{ for any } h_0 \in C^\infty(M_0)] \tag{4}$$

$$j^*(i(X)\Omega - \beta) = 0 \qquad [\text{locally } \beta = dh \text{ for any } h \in C^\infty(P)]$$

We say then that (M_0, ω_0, X_0) [or (P, Ω, X)] are *weak, locally Hamiltonian systems* (wlHs) relative to C, and X_C is a *locally Hamiltonian field* in C.

This way of defining the dynamics in C is equivalent to the one given by Lichnerowicz (1975). In addition, equations (3) allow us to formulate a theory of canonical transformations for presymplectic systems (Gomis *et al.*, 1984; Cariñena *et al.*, 1985). This is due to the essential fact that these dynamics are just those leaving invariant the geometrical structure ω_C in C. In fact:

Theorem. A necessary and sufficient condition for (M_0, ω_0, X_0) [or, when appropriate, (P, Ω, X)] being a wlHs relative to C is that ω_C be an absolute invariant form under X_C; that is,

$$0 = L(X_C)\omega_C = \hat{\imath}^*(L(X_0)\omega_0) = j^*(L(X)\Omega)$$

In turn, this condition is equivalent to demanding that the *Poincaré integral* associated to X_C be invariant under X_C; that is, if $D \subset C$ is a 2-domain of integration and $\{F_t\}$ denotes the uniparametric local group of diffeomorphisms generated by X_C, we have

$$\frac{d}{dt}\int_{F_t(D)} \omega_C = 0$$

What this theorem means is that equations (3) may be obtained from a variational principle [for equations (2) this is not possible]. So, the preceding result generalizes previous results in a geometrical way (Benavent and Gomis, 1979; Dominici and Gomis, 1980, 1982).

Next we want to study the relation between both kinds of dynamics in C: the induced one (2) and the geometrical ones (4). A first conclusion is that, taking any β_0 such that $\hat{\imath}^*\beta_0 = \hat{\imath}^*\alpha_0$, then every $X_0 \in \chi(M_0)$ solution of (2) is also a solution of (4), and therefore (2) is a particular case of (4). On the other hand, if we analyze the gauge content of the dynamical theory given by (4), we have that all the gvf are known from the beginning and exhaust $\underline{\mathrm{Ker}\,\omega_C}$ (we denote by $\underline{\mathrm{Ker}\,\omega_C}$ an extension to M_0 of $\mathrm{Ker}\,\omega_C$; that is, we have $\hat{\imath}_*\,\mathrm{Ker}\,\omega_C = \underline{\mathrm{Ker}\,\omega_C}|_C$), which is the gauge group obtained in the canonical transformations theory for constrained systems (Cariñena *et al.*, 1985). On the contrary, the set G of gvf of the dynamics (2) is such that $G \subset \underline{\mathrm{Ker}\,\omega_C}$.

It would be very interesting if $G = \underline{\mathrm{Ker}\,\omega_C}$, because in such a case we would have assured that: (a) The induced dynamics (2) is completely equivalent to any of the natural ones (4), in particular to the one satisfying $\beta_C = \hat{\imath}^*\alpha_0$; (b) the gauge degrees of freedom and the degrees of degeneracy of the presymplectic structure of C are identical.

This problem has been solved by Gotay and Nester (1978), who prove that, in the Hamiltonian formalism, both sets coincide if and only if every first-class secondary constraint ζ is an effective constraint, that is, it satisfies $d\zeta|_C \neq 0$.

At this point there are some important questions to be pointed out. In relation to comment (a), note that the equivalence between the dynamics (2) and (4) implies the validity of Dirac's conjecture (Dirac, 1964) and conversely. Therefore, the acceptance of this conjecture is justified whenever the condition of Gotay and Nester is satisfied. On the other hand, and related to comment (b), it is known that a way to eliminate the degeneraracy of the presymplectic structure ω_C is to go to the *reduced phase space* (rps) (Lichnerowicz, 1975), which is obtained from (C, ω_C) by performing the quotient of C by the foliation generated by the involutive distribution $\mathrm{Ker}\,\omega_C$. Then, if we have assured the identification $G = \underline{\mathrm{Ker}\,\omega_C}$, this procedure allows one to eliminate both degeneracy and the physically irrelevant gauge freedom.

Summing up, we can conclude that the dynamics induced by a plHs on its fcs is equivalent to any of the geometrically natural ones that can be defined in it by imposing the invariance of the Poincaré integral if, and only if, Dirac's conjecture is accepted. (In turn, the validity of this conjecture is equivalent to the nonexistence of ineffective first-class secondary constraints; this means that, if they appear, they must be substituted by equivalent effective ones). In addition, in this way, the reduction procedure to the rps eliminates degeneracy and gauge freedom at the same time.

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REFERENCES

Benavent, F., and Gomis, J. (1979). *Annals of Physics*, **118**, 476.

Bergvelt, M. J., and De Kerf, E. A. (1986). *Physica*, **139A**, 101-148.

Cariñena, J. F., Gomis, J., Ibort, L. A., and Román, N. (1985). *Journal of Mathematical Physics*, **26**(8), 1961.

Dirac, P. A. M. (1964). *Lectures in Quantum Mechanics*, Belfer Graduate School of Science, Yeshiva University New York.

Dominici, D., and Gomis, J. (1980). *Journal of Mathematical Physics*, **21**, 2124.

Dominici, D., and Gomis, J. (1982). *Journal of Mathematical Physics*, **23**, 256.

Gomis, J., Llosa, J., and Román, N. (1984). *Journal of Mathematical Physics*, **25**(5), 1348.

Gotay, M. J. (1982). *Proceedings of the American Mathematical Society* **84**, 111.

Gotay, M. J., and Nester, J. M. (1979). Presymplectic Hamiltonian and Lagrangian system, gauge Transformations and Dirac's theory of constraints, in *Lecture Notes in Physics*, Vol. 94, p. 272, Springer, Berlin.

Gotay, M. J., Nester, J. M. and Hinds, G. (1978). *Journal of Mathematical Physics*, **19**(11), 2386.

Lichnerowicz, A. (1975). *Comptes Rendus de l'Academie des Sciences (Paris)*, **280A**, 523.

Marle, C. M. (1983). *Asterisque*, **107-108**, 69.

Sniatycki, J. (1974). *Annales de l'Institut Henri Poincaré*, **20**, 365.